Real Analysis — Exam 2

Instructions

This exam consists of two parts, an In Class Portion and a Take Home Portion.

The In Class Portion will be a closed-book, closed-notes, timed exam. It consists of 15 questions, and answers need not be justified or proved (unless explicit instructions indicate otherwise). However, brief descriptions of ideas and concepts may be worth partial credit.

The Take Home Portion consists of six questions. You are required to complete questions 1-4 (totaling 60 points), and you may choose any two problems from questions 5-14 (totaling at least 40 points). For this portion of the exam, you may consult your notes, our class website, and Chapters 0, 1, 2 and 3 from our textbook, but no other materials may be used. In particular, no other websites, textbooks, or notes can be used except those already mentioned; please note that you are not allowed to use material from Chapters 4, 5, 6 or 7 from our textbook. You are not allowed to discuss this exam with anyone except me, and I encourage you to do so!

Also note that, unless otherwise stated, the set $\mathbb{R}$ with no specified metric will refer to the metric space $(\mathbb{R}, \text{standard metric})$ where distances are measured using $|x - y|$.

In Class Portion

1. A sequence of points $\{a_n\}$ in $\mathbb{R}$ is said to converge to $a \in \mathbb{R}$ if ... (complete the precise definition)

2. A sequence of points $\{a_n\}$ in $\mathbb{R}$ is said to be Cauchy if (complete the precise definition)

3. A function $f : D \to \mathbb{R}$ is said to have a limit L at $x_0 \in D$ if ... (complete the precise definition)

4. A function $f : D \to \mathbb{R}$ is said to be continuous at $x_0 \in D$ if ... (complete the precise definition)

5. A function $f : D \to \mathbb{R}$ is said to be uniformly continuous (on D) if ... (complete the precise definition)

6. A set $S \subseteq \mathbb{R}$ is said to be open if ... (complete the precise definition)

7. A set $S \subset \mathbb{R}$ is said to be closed if... (complete the precise definition)

8. An open cover for a set $S \subseteq \mathbb{R}$ is ... (complete the precise definition)

9. A set $S \subseteq \mathbb{R}$ is said to be compact if ... (complete as many definitions as you can recall; these additional answers will be worth bonus points)

10. Provide an example of a subset $S \subset \mathbb{R}$ that is connected, and provide an example of a subset $T \subseteq \mathbb{R}$ that is disconnected.

11. True or false: It is impossible for a monotone function $f : [0, 2] \to \mathbb{R}$ to be discontinuous at every point in the subset $[1/2, 1]$.

12. Given that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$f(x) = x^2 \text{ for every } x \in \mathbb{Q}$$

compute the value of $f(\pi/2)$.

13. State the Bolzano-Weierstrass Theorem.

14. State the Heine-Borel Theorem.

15. Suppose we have two real numbers, $A, B \in \mathbb{R}$, and we know that for every $\varepsilon > 0$ it follows that $d(A, B) = |A - B| < \varepsilon$. What can we conclude about A and B?

1 Take Home Portion : Mandatory Questions

1. (10 points) Prove that $f(x) = \sqrt{x}$ is continuous for all $x \geq 0$.

2. (25 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies the following property: there exists a constant $c \in (0, 1)$ so that
$$|f(x) - f(y)| \leq c\,|x - y|$$
for every $x, y \in \mathbb{R}$. Such a function is called a **shrinky-dinky mapping**.

   (a) (5 points) Prove that f is continuous at every point $x_0 \in \mathbb{R}$.

   (b) (2.5 points) Is f uniformly continuous on $\mathbb{R}$? Prove your answer.

   (c) (10 points) Let $x_1 \in \mathbb{R}$ be a fixed real number. Prove that the recursively defined sequence
$$\{x_n\}_{n=1}^{\infty} \text{ where } x_{n+1} = f(x_n)$$
   is a Cauchy sequence.

   (d) (2.5 points) Explain why the limit of the sequence from part (c),
$$x_0 = \lim_{n\to\infty} x_n,$$
   is a fixed point of f, i.e. satisfies $f(x_0) = x_0$.

   (e) (5 points) Prove that if y_0 is another fixed point for f, then $x_0 = y_0$.

   (f) (ε points) Congratulate yourself on proving a very famous and important theorem!

(+10) Rephrase the question and prove parts (a)-(f) above for an arbitrary, complete metric space (X, d). This first requires you to define the term **shrinky-dinky mapping** for a function $f : X \to X$.

3. (15 points) For this problem we first define what it means for a set to be **Sandy**. We say that a set $S \subseteq \mathbb{R}$ is a **Sandy** set if for each point $x \in S$ there exists $\delta > 0$ so that

$$B_\delta(x) \cap S = (x - \delta, x + \delta) \cap S = \{x\}.$$

   (a) (5 points) Provide an example of a non-empty Sandy set, $S \subseteq \mathbb{R}$.

   (b) (7 points) Prove that if a set $S \subseteq \mathbb{R}$ is both Sandy and compact, then S is finite.

   (c) (3 points) Is the converse to part (b) true or false? Prove your answer.

4. (10 points) On a separate sheet of paper (one that can be stapled to the rest of your exam), please share some of the ways in which Dr. Sandy Ganzell has helped make your SMCM math experience a positive one. I intend to share these responses with him, so if you would like to remain anonymous you may do so by withholding your name and any identifying information. Lastly, this is the one exam question on which you may collaborate. If you have not had a class with Sandy, then please do yourself a favor and chat with some of the other people who have, then discuss their thoughts and opinions in your response.

Take Home Exam – Choose any two of the following

5. (20 points) For both parts of this problem, suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

   (a) (15 points) Suppose $C \subseteq \mathbb{R}$ is a closed subset. Prove that the pre-image

   $$f^{-1}(C) = \{x \in \mathbb{R} : f(x) \in C\}$$

   is also a closed set.

   (b) (5 points) Is it true that if $K \subseteq \mathbb{R}$ is a compact subset then the pre-image $f^{-1}(K)$ is also compact? If so, provide a proof. If not, provide a counterexample.

6. (20 points) A function $f : D \to \mathbb{R}$ is called **Meadows** if there exists $M > 0$ so that

   $$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

   for every $x, y \in D$. (The constant M is called "the Meadows bound.")

   (a) (15 points) Prove that every Meadows function is uniformly continuous.

   (b) (5 points) Prove that the converse to part (a) is not true.

   Bonus (+10): Do part (a) from the problem above, replacing $\mathbb{R}$ with an arbitrary metric space (X, d).

7. (20 points) This problem is all about closed subsets of $\mathbb{R}$.

   (a) (5 points) Prove that the union of finitely many closed sets $F_1, F_2, \cdots, F_n \subseteq \mathbb{R}$ is a closed set.

   (b) (5 points) Prove that the intersection of an arbitrary family of closed sets is a closed set. In other words, prove that if $\{F_\lambda : \lambda \in \Lambda\}$ is a collection of closed sets, then the intersection

   $$\bigcap_{\lambda \in \Lambda} F_\lambda$$

   is also closed.

   (c) (10 points) Prove that if $K \subseteq \mathbb{R}$ is compact and if $F \subseteq \mathbb{R}$ is closed, then the set $K \cap, F$ is also closed. Is this set also compact? Prove your answer.

8. (20 points) Prove or disprove: there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that, for each $c \in \mathbb{R}$, the equation $f(x) = c$ has exactly two solutions.

9. (20 points) This problem is all about open subsets of $\mathbb{R}$.

   (a) (8 points) Prove that if $U \subseteq \mathbb{R}$ is open and $F \subseteq \mathbb{R}$ is an arbitrary finite set, then the set $U \setminus F$ is also open. Is this true for countable subsets F? Prove your answer or provide a counterexample.

   (b) (7 points) Let $S \subseteq \mathbb{R}$ be a subset, and define **the interior of** S to be the set

   $$S^\circ = \{x \in S : x \text{ is an interior point for } S\}.$$

   Prove that S° is always open.

   (c) (5 points) Given a set $S \subseteq \mathbb{R}$ is it true that

   $$\overline{(S^\circ)} = S\,?$$

   If so, prove your answer. If not, provide a counterexample.

10. A function $f : D \to \mathbb{R}$ is **locally bounded** on $D \subseteq \mathbb{R}$ if for each $c \in D$ there exists $\delta > 0$ such that the set

    $$\{f(x) : x \in (c-\delta, c+\delta) \cap D\}$$

    is bounded.

    (a) (10 points) Suppose $f : [0,1] \to \mathbb{R}$ is locally bounded on the compact interval $[0,1]$. Prove that f is bounded on the entire interval $[0,1]$.

    (b) (10 points) Give an example of a function $f : (0,1) \to \mathbb{R}$ that is locally bounded on $(0,1)$ but is not bounded on $(0,1)$.

11. (20 points) Let $K_n \subseteq \mathbb{R}$ be a nested sequence of non-empty, compact sets – that is, suppose $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots K_n \supseteq K_{n+1} \supseteq \cdots$. Prove that

    $$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

12. (20 points) Let $f, g : D \to \mathbb{R}$ be uniformly continuous. Prove that $f+g : D \to \mathbb{R}$ is uniformly continuous. What can be said about $f \cdot g$? Justify your answer.

13. (20 points) For this problem we will use the metric space (X, d) where $X = \mathbb{R}$ but the metric d is given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(a) (7 points) Prove that every set $S \subseteq X$ is an open set. Explain why this implies that every set $T \subseteq X$ is also closed.

(b) (12 points) Prove that a set $S \subseteq X$ is compact (in the "open cover" sense of the word) if and only if S is finite.

(c) (1 point) Is the Heine-Borel Theorem true in the metric space (X, d)?

14. (25 points) For this problem we will use two definitions. First, a set $P \subseteq \mathbb{R}$ is said to be a **perfect set** if P is closed and if $P \subseteq P'$ (i.e. every point is an accumulation point of P).

Second, given a set $S \subseteq \mathbb{R}$ we say a point $x \in \mathbb{R}$ is a **condensation point** for S if every neighborhood of x intersects S in an uncountable set. That is, x is a **condensation point** if for every $\delta > 0$ it follows that

$$B_\delta(x) \cap S = (x - \delta, x + \delta) \cap S \text{ is uncountable .}$$

Lastly, we will notate the set of condensation points of a given set S as

$$S^{\clubsuit} = \{x \in \mathbb{R} : x \text{ is a condensation point for} S\} .$$

(a) (5 points) Given a set $S \subseteq \mathbb{R}$, prove that $S^{\clubsuit} \subseteq S'$.

(b) (15 points) Given a set $S \subseteq \mathbb{R}$, prove that $S^{\clubsuit}$ is perfect.

(c) (5 points) Find an example of a set $S \subseteq \mathbb{R}$ where $S^{\clubsuit} \neq S'$.

Bonus Question (+5 but +10 for Nick M.) Let us call a set $S \subseteq \mathbb{R}$ **condensed** if $S^{\clubsuit} \subseteq S$, and define the set

$$\textbf{condensed-}S = S \cup S^{\clubsuit}.$$

Given the set

$$\text{milk} = (0, 1)$$

explain why the set $[0, 1]$ is so much sweeter!