Exam 1

Instructions: This exam consists of 6 questions. Credit will be awarded for correct, well written solutions in which all work has (clearly) been shown. If you have any questions, feel free to ask me.

(1) (21 points) Several statements are written below. Determine which ones are true and which ones are false. If you think a statement is false explain why.

(a) The equation $5x + 2y - z = 0$ has as its solution set a plane in $\mathbb{R}^3$ that passes through the point $\vec{x}_0 = (1, 1, 12)$ with normal $\vec{n} = (5, 2, -1)$.

(b) The cross product of two vectors $\vec{v}, \vec{w}$ only makes sense if $\vec{v}, \vec{w} \in \mathbb{R}^3$.

(c) The function $\vec{x} : \mathbb{R} \to \mathbb{R}^2$ given by $\vec{x}(t) = (1 - t, 2)$ parameterizes the line in $\mathbb{R}^2$ that passes through the point $\vec{x}_0 = (1, 2)$ and is parallel to the vector $\vec{v} = (-1, 0)$.

(d) The vectors $\vec{v} = (2, 4, 2)$ and $\vec{w} = (1, 1, -5)$ are orthogonal.

(e) The volume of the parallelepiped spanned by the vectors $\vec{v}, \vec{w}$, and $\vec{u}$ (each in $\mathbb{R}^3$) is given by

$$V = |(\vec{u} \cdot \vec{v}) \times \vec{w}| \, .$$

(f) The point $(1, 1, 1) \in \mathbb{R}^3$ has cylindrical coordinates $r = \sqrt{2}, \theta = \pi$, and $z = 1$.

(g) To convert from spherical coordinates (ρ, ϕ, θ) to rectangular coordinates, (x, y, z), one can use the equations

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \cos\theta \\ z &= \rho \cos\phi \end{aligned}$$

(2) (19 points) Given three vectors $\vec{u} = (1, 2, 3), \vec{v} = (-1, 0, 1)$ and $\vec{w} = (0, 1, 0)$, calculate

(a) (3 points) $\vec{u} \cdot \vec{v}$

(b) (3 points) the magnitude of $\vec{w}$

(c) (3 points) the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$

(d) (10 points) the equation for the plane containing all three points / position vectors $\vec{u}, \vec{v}$, and $\vec{w}$.

(3) (20 points) Consider the equation (expressed in cylindrical coordinates)

$$z^2 = r^2.$$

(a) (2 points) Rewrite this equation in rectangular coordinates.

(b) (8 points) Sketch a picture of the surface determined by this equation. In your picture indicate the sections $z = -1, z = 0, z = 1$, and $x = 0$.

(c) (5 points) Tom Riddle claims that the surface determined by the equations (expressed in spherical coordinates)

$$\phi = \pi/4 \text{ or } \phi = 3\pi/4$$

coincides with the surface you sketched in part (b). Is he right? How do you know?

(d) (3 points) Is the surface you sketched the graph of a function?

(e) (2 points) At what points (if any) does the line $\vec{r}(t) = (3t, 4t, 5t)$ intersect the surface you sketched in part (b)?

Bonus (+5) Rewrite the equation you obtained in part (a) of this problem so that it has for the form of a level set $F(x, y, z) = 0$. For five bonus points, compute the partial derivatives $\partial F/\partial x, \partial F/\partial y$, and $\partial F/\partial z$.

(4) (20 points)

(a) (10 points) Dr. Heather Moon claims that the function $\vec{x} : \mathbb{R} \to \mathbb{R}^2$ given by

$$\vec{x}(t) = (2\cos t, 2\sin t)$$

parameterizes a line. However, Dr. Emek Köse disagrees; she claims that $\vec{x}$ parameterizes a circle. Who is correct? Explain your answer; if Dr. Moon is correct, be sure to specify a point through which the line passes and a direction in which it heads. If Dr. Köse is correct, be sure to specify the point at which the circle is centered and its radius.

(b) (10 points) The surface determined by the equation $r = 2$ is a vertical cylinder in $\mathbb{R}^3$. Find a parameterization, i.e. a function $\vec{x} : \mathbb{R}^2 \to \mathbb{R}^3$, of this surface.

Bonus (+5) What curve does the function $\vec{x}(t) = (2\cos t, 2\sin t, 2t)$ parameterize? (Sketch a picture.)

Bonus (+5) What surface does the function

$$\vec{x}(u, v) = (\sin u \cos v, \sin u \sin v, 5\cos u)$$

parameterize? (Sketch a picture.)

(5) (20 points) For this problem, use the function $f(x, y) = e^{y-\sin x}$.

(a) (3 points) Determine the values of n and m that make the following notation true:

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

(b) (7 points) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$

(c) (10 points) Sketch and describe the level sets where $f(x, y) = 1$ and where $f(x, y) = e^2$.