Real Analysis — Homework 4 Solutions

DUE WEDNESDAY, OCTOBER 5

Reading and Notes Questions

1. State the official definition of the sentence "f has a limit L at x_0" that appears in our textbook.

   Answer: Look it up!

2. Rephrase the definition from Question 1. using the language and notation of "ε-balls" and "punctured δ-balls."

   Answer: We say that f has a limit L at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

   $$\text{if } x \in D \cap B_\delta^*(x_0), \text{ then } f(x) \in B_\varepsilon(L).$$

   Here $B_\delta^*(x_0)$ is a "punctured δ-ball" centered at x_0 and is defined as

3. If we write $\lim_{x \to x_0} f(x) = L$, then what (if anything) can we conclude about $f(x_0)$?

   Answer: Nothing! $f(x_0)$ may not even exist! It may equal a number, but that number may or may not equal the limit value, L. Indeed, when $f(x_0) = L = \lim_{x \to x_0} f(x)$, then f is said to be **continuous** at x_0.

4. On pages 65-66, our textbook explores the function $f(x) = x/|x|$. Does their discussion prove that the limit of f(x) at x = 0 exists or doesnt exist? What style of proof did they use in this discussion?

   Answer: They prove that the limit does not exist, and they do so by contradiction. Note that what they prove is that there cannot ever exist *any* number $L \in \mathbb{R}$ that satisfies $L = \lim_{x \to 0} f(x)$. This is a pretty serious thing to prove, and so it kind of makes sense it would be handled by contradiction (some sort of direct proof might require us to test and throw out *every possible real number* as a potential value for the limit).

5. In the textbooks discussion of Example 2.5 (pages 68-69), the line "However, for a fixed positive integer q, there are but a finite number of points in [0, 1] of the form p/q. In fact, for a fixed positive number q_0, there are only a finite number of points in [0, 1] of the form p/q where $q \leq q_0$ with p and q positive integers."

   Explain why this line is true. Also, the function being discussed has a name, it is _____'s function. Find the name!

   Answer: I'm skipping this one, except for the name part. This is Thomae's function.

6. Draw a visual interpretation of Theorem 2.1 (on page 69). Why in the statement of this theorem is it stipulated that $x_n \neq x_0$? In the discussion before the proof of this theorem, what previous homework problem was used?

   Answer: skipped!

7. Write a very short (but rigorous!) proof of Theorem 2.2 (you may, of course, assume Theorem 2.1 is true).

   Answer: Proof. This follows immediately from Theorem 2.1 and from the fact that every Cauchy sequence converges.

8. What is a less formal way to express the content of Theorem 2.3?

   Answer: If f has a limit at x_0, then f is bounded near x_0.

9. Write a poem about Example 2.6.

   Answer: nope!

10. Explain exactly where Theorem 2.4 was used in Example 2.7.

11. In the definition of "limit of a function" (on page 64), is our textbook actually claiming that x_0 is an interior point for the domain D of f? Explain your answer.

1 Proof Questions

1. Prove or disprove: The set $\mathbb{Q} \subset \mathbb{R}$ is open.

Claim The set of rationals is not an open subset of the real numbers.

proof. To prove that $\mathbb{Q}$ is not open, we need to argue that it contains one point that is not interior to $\mathbb{Q}$. Indeed, something much stronger can be shown: no point of $\mathbb{Q}$ is an interior point! (So, $\mathbb{Q}$ is, like, super-not-open.)

To prove this, let $q \in \mathbb{Q}$ be arbitrary, and let $\delta > 0$ be arbitrary, too. By a previous theorem (which one, exactly??) we know that between any two real numbers there exists an irrational number. In particular, between the two real numbers q and $q - \varepsilon$ there exists $r \in \mathbb{R} \setminus \mathbb{Q}$. In other words, we have that for every $q \in \mathbb{Q}$ and for every $\delta > 0$

$$B_\delta(q) \bigcap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$$

and so

$$B_\delta(q) \not\subseteq \mathbb{Q}.$$

Hence, $\mathbb{Q}$ is super-not-open. $\square$

Secret Bonus: we can also prove that $\mathbb{Q}$ is not closed by proving that the set of irrationals is similarly not open. Indeed, one can also prove that no point of the irrationals is an interior point!

2. (a) Complete the following proof: Suppose $f : D \to \mathbb{R}$ is a function with x_0 an accumulation point of D, and assume L_1 and L_2 are limits of f at x_0 (in accordance with the definition on page 64).

   Let $\varepsilon > 0$ be given. We aim to show that $|L_1 - L_2| < \varepsilon$, which will complete our proof. By definition of limit, there exists a ??? > 0 so that

$$0 < |x - x_0| < ??? \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{??}$$

$$0 < |x - x_0| < ??? \Rightarrow |f(x) - L_2| < \frac{\varepsilon}{??}$$

   (assuming $x \in D$, too). Let such a ???? > 0 be chosen, and let $x \in D$ and $x \in B_{??}(x_0)$. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$$

$$\leq |L_1 - f(x)| + |L_2 - f(x)| < \frac{\varepsilon}{??} + \frac{\varepsilon}{??} = \varepsilon$$

   as desired. Therefore, L_1 _____ L_2. □

   (b) This proof should feel suspiciously similar to the proof for a previous homework problem. Which one was/is it?

   (c) Is it possible for a function $f : D \to \mathbb{R}$ have two different limits at a single point x_0?

3. Complete exercise 11. on page 79. (skipped for now)

4. Complete exercise 16. on page 80 (citing relevant theorems from Section 2.3). (skipped for now)

5. Prove that any function of the form $f(x) = mx+b$ where $m, b \in \mathbb{R}$ are constants is continuous at every point in $\mathbb{R}$.

proof. Let $m, b \in \mathbb{R}$ be given, and consider the function $f(x) = mx + b$. First note that the domain for this function is $D = \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be arbitrary. We want to prove that

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} mx + b = mx_0 + b = f(x_0).$$

To this end let $\varepsilon > 0$ be given.

Case 1. If $m = 0$ then the function under consideration is constant, i.e. $f(x) = b$ and so $f(x_0) = b$. In this case we can choose δ to be any positive number. It then follows that whenever $x \in B^*_\delta(x_0)$,

$$d(f(x), f(x_0)) = |f(x) - f(x_0)| = |b - b| = 0 < \varepsilon$$

.

Case 2. Suppose $m \neq 0$. In this case choose $\delta = \varepsilon/|m|$. If $x \in B^*_\delta(x_0)$, then

$$\begin{aligned} d\left(f(x), f(x_0)\right) &= |mx + b - (mx_0 + b)| = |m(x - x_0)| \\ &= |m|\ |x - x_0| < |m|\delta = |m|\frac{\varepsilon}{|m|} = \varepsilon. \end{aligned}$$

Therefore, at every $x_0 \in \mathbb{R}$, the limit of f is the value $f(x_0)$, satisfying the definition for "continuity at every point in $\mathbb{R}$." □

6. Prove that the set S = (-10, 0] is neither open nor closed.

proof. First, to show that S is not open we must show that at least one of its elements fails to be an interior point. In fact, this set only has one non-interior-point, the element $0 \in S$. Let $\delta > 0$ be any positive number. The δ-ball $B_\delta(0)$ contains points that are not elements of S since

$$S = \{x \in \mathbb{R} : -10 < x \leq 0\} \text{ and } B_\delta(0) = \{x \in \mathbb{R} : d(x, 0) < \delta\} = (-\delta, \delta).$$

In particular, the point $\delta/2 \in B_\delta(0)$ but $\delta/2 \notin S$ since $\delta/2 > 0$. Therefore, no δ-ball centered at 0 is contained in S, and so 0 is not an interior point.

To prove that S is not closed, we must prove that the complement $\mathbb{R} \setminus S$ is not open. Fortunately, we can write

$$\mathbb{R} \setminus S = (-\infty, 10] \bigcup (0, \infty)$$

and so our proof will be complete if we can show that this set contains a point that is not an interior point. Using an argument like the one deployed above for the point $0 \in S$, one can show that the point $10 \in \mathbb{R} \setminus S$ fails to be an interior point. □